

On some properties of orthogonal Weingarten functions

Benoît Collins and Sho Matsumoto

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Abstract

We give a Fourier-type formula for computing the orthogonal Weingarten formula. The Weingarten calculus was introduced as a systematic method to compute integrals of polynomials with respect to Haar measure over classical groups. Although a Fourier-type formula was known in the unitary case, the orthogonal counterpart was not known. It relies on the Jack polynomial generalization of both Schur and zonal polynomials. This formula substantially reduces the complexity involved in the computation of Weingarten formulas. We also describe a few more new properties of the Weingarten formula, state a conjecture and give a table of values.

1 Introduction

The terminology of Weingarten calculus was introduced in [Co], and designates the formula involved in the computation of the integral of polynomial functions over compact matrix groups with respect to their Haar measure. Weingarten calculus is an analogy of Wick calculus, which tells that the joint moments of random vectors of a real Gaussian space V can be computed by the formula

$$E(v_1 \dots v_{2n}) = \sum_{\mathbf{m} \in \mathcal{M}(2n)} E(v_{\mathbf{m}(1)} v_{\mathbf{m}(2)}) \dots E(v_{\mathbf{m}(2k-1)} v_{\mathbf{m}(2k)}),$$

where v_i are gaussian vectors of V , E is the expectation, and $\mathcal{M}(2n)$ is the set of pair partitions of $\{1, \dots, 2n\}$ (see section 2 for the notations and definitions). This formula covers the computation of all moments, as odd moments are zero due to the $1 \rightarrow -1$ symmetry.

The question of systematically computing the Haar measure on polynomials originally emanates from theoretical physics. One of the most notable attempts is due to Weingarten in [Wei], and this is why this calculus bears his name. In the paper [Co], only the case of integration over the unitary group is being considered, with a restriction on the bound of the polynomial. This restriction is relieved later in the paper [CSn], where an orthogonal version of the Weingarten was developed.

However, whereas in [Co], an explicit Fourier type formula for the unitary Weingarten function is given as a sum on the irreducible summands of the n^{th} tensor power of the fundamental representation of the unitary group, we were not able to do so in a satisfactory way for the orthogonal group case, although the canonical description of the irreducible summands of the

n^{th} tensor power of the fundamental representation of the orthogonal group is very similar to the unitary group case.

Our present paper addresses this question. The solution comes through the observation that the Schur polynomials involved in the Fourier-type formula for the Weingarten formula in the unitary case are actually a special case of so called Jack polynomials.

The organization of this paper is as follows: in section 2, we recall the Weingarten formula in the orthogonal case. For the sake of completeness we give a complete proof, adapted from [BCo]. This proof is much simpler than the original one. In section 3, we state our main result and then prove it in section 4. In section 5, we consider as an application the example of a truncated orthogonal random orthogonal matrix. In section 6, we study the specific case of the orthogonal Weingarten function evaluated pairings that form together a full cycle. This leads us to some new properties of the orthogonal Weingarten functions, and to a conjecture. We finish with section 7, where we give a table of values of the Weingarten function.

We would like to point out that the case of the symplectic group can be treated in a similar fashion. We intend to deal with that case in a forthcoming paper.

2 Weingarten convolution formula for orthogonal and unitary groups

In this section we recall the Weingarten formula for orthogonal groups. We finish this section by recalling the unitary formula. Chronologically, the unitary formula was established earlier, but our approach makes it simpler to consider the orthogonal case first.

Let $\mathcal{M}(2n)$ be the set of all pair partitions on $\{1, 2, \dots, 2n\}$. Each pair partition \mathbf{m} in $\mathcal{M}(2n)$ is uniquely expressed by the form

$$(2.1) \quad \{\{\mathbf{m}(1), \mathbf{m}(2)\}, \{\mathbf{m}(3), \mathbf{m}(4)\}, \dots, \{\mathbf{m}(2n-1), \mathbf{m}(2n)\}\}$$

with $\mathbf{m}(2i-1) < \mathbf{m}(2i)$ for $1 \leq i \leq n$ and with $\mathbf{m}(1) < \mathbf{m}(3) < \dots < \mathbf{m}(2n-1)$.

Given two pair partitions $\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)$, we define the graph $\Gamma(\mathbf{m}, \mathbf{n})$ as follows. The vertex set of $\Gamma(\mathbf{m}, \mathbf{n})$ is $\{1, 2, \dots, 2n\}$ and the edge set consists of $\{\mathbf{m}(2i-1), \mathbf{m}(2i)\}$ and $\{\mathbf{n}(2i-1), \mathbf{n}(2i)\}$ with $1 \leq i \leq n$. The *Gram matrix* $G_n^{O(d)} = (G^{O(d)}(\mathbf{m}, \mathbf{n}))_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)}$ is given by

$$G^{O(d)}(\mathbf{m}, \mathbf{n}) = d^{\text{loop}(\mathbf{m}, \mathbf{n})},$$

where $\text{loop}(\mathbf{m}, \mathbf{n})$ is the number of loops in the graph $\Gamma(\mathbf{m}, \mathbf{n})$, i.e., the number of connected components of $\Gamma(\mathbf{m}, \mathbf{n})$.

Let $\text{Wg}_n^{O(d)} = (\text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n}))_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)}$ be the pseudo-inverse matrix of $G_n^{O(d)}$. We recall that for any real symmetric matrix A there exists a unique matrix B satisfying $ABA = A$ and $BAB = B$. If A is invertible, then $B = A^{-1}$. The matrix B is called the *pseudo-inverse* of A . We call $\text{Wg}_n^{O(d)}$ the *Weingarten matrix* for the orthogonal group. The following Theorem was proved in [CSn]:

Theorem 2.1 ([CSn]). *Given indices i_1, \dots, i_{2n} and j_1, \dots, j_{2n} in $\{1, 2, \dots, d\}$,*

$$(2.2) \quad \int_{g \in O(d)} g_{i_1 j_1} \cdots g_{i_{2n} j_{2n}} dg = \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)} \text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n}) \prod_{k=1}^n \delta_{i_{\mathbf{m}(2k-1)}, i_{\mathbf{m}(2k)}} \delta_{j_{\mathbf{n}(2k-1)}, j_{\mathbf{n}(2k)}}.$$

Observe also that in the odd case, we always have $\int_{g \in O(d)} g_{i_1 j_1} \cdots g_{i_{2n+1} j_{2n+1}} dg = 0$ as the Haar measure is invariant under the transformation $1 \rightarrow -1$. Let us illustrate this Theorem by a few examples:

Example 2.1. • Integrands of type $g_{i_1 j_1} \cdots g_{i_{2n} j_{2n}}$ which don't have pairings integrate to zero. For example, $\int_{g \in O(d)} g_{11}^3 g_{12} dg = \int_{g \in O(d)} g_{11}^3 g_{22} dg = 0$.

- The integral below admits only one pairing ($1 \rightarrow 1$ and $2 \rightarrow 2$). Therefore,

$$\int_{g \in O(d)} g_{11}^2 g_{22}^2 dg = \text{Wg}^{O(d)}(\{\{1, 2\}, \{3, 4\}\}, \{\{1, 2\}, \{3, 4\}\}).$$

A direct computation shows that this coefficient of $\text{Wg}_4^{O(d)}$ is $\frac{d+1}{d(d+2)(d-1)}$ (alternatively, see Section 7). Therefore

$$\int_{g \in O(d)} g_{11}^2 g_{22}^2 dg = \frac{d+1}{d(d+2)(d-1)}$$

for any $d \geq 2$.

- For the integral $\int_{g \in O(d)} g_{11}^4 dg$, all pairings are admissible (because all the indexes are the same, equal to 1). Therefore the integral is the sum of all entries of $\text{Wg}_4^{O(d)}$. A direct computation shows that the answer is $\frac{3}{d(d+2)}$ for all d . Observe that this is indeed the second moment of a hyperspherical distribution.

Consider the vector space where $O(d)$ acts, namely

$$V = \mathbb{C}^d$$

and denote by e_1, \dots, e_d its standard basis. We need to introduce an embedding of $\mathcal{M}(2n)$ in $V^{\otimes 2n}$, which we shall denote by ρ . This embedding is defined by the formula

$$\rho(\mathbf{m}) = \sum_{j_1 \dots j_{2n}} \binom{\mathbf{m}}{j_1 \dots j_{2n}} e_{j_1} \otimes \dots \otimes e_{j_{2n}},$$

where the symbol $\binom{\mathbf{m}}{j_1 \dots j_{2n}}$ here is considered as a generalized Kronecker delta symbol: it is 1 if all strings of \mathbf{m} join pairs of equal indices, and is 0 if not.

Example 2.2. If $\mathbf{m} = \cap \in \mathcal{M}(2)$, then

$$\rho(\mathbf{m}) = \sum_{j_1, j_2} \binom{\cap}{j_1, j_2} e_{j_1} \otimes e_{j_2} = \sum_{j_1, j_2} \delta_{j_1, j_2} e_{j_1} \otimes e_{j_2} = \sum_{j=1}^d e_j \otimes e_j.$$

Although we won't need this fact, let us remark that ρ is an action in a Brauer algebra sense. We will also recover in the course of the proof of Theorem 3.1, the fact that the family $\{\rho(\mathbf{m}), \mathbf{m} \in \mathcal{M}(2n)\}$ is an independent family provided that $d \geq n$.

We denote by $\text{Fix}_{O(d)} V^{\otimes n}$ the vector subspace of $V^{\otimes n}$ of invariant elements under the action of $O(d)$, namely, elements x that satisfy for all $g \in O(d)$, $g^{\otimes n} \cdot x = x$. The main ingredient for the proof of Theorem 2.1 is the following classical result, sometimes known as *orthogonal Schur-Weyl duality* formula. We refer to [GWA] for a proof and details.

Proposition 2.2 (see, e.g. [GWA]). *In the odd case, we have $\text{Fix}_{O(d)}(V^{\otimes 2n+1}) = \{0\}$. In the even case, we have*

$$\text{Fix}_{O(d)}(V^{\otimes 2n}) = \text{span}\{\rho(\mathfrak{m}), \mathfrak{m} \in \mathcal{M}(2n)\}.$$

The above proposition is the main ingredient to the following lemma:

Lemma 2.3. *The element $P = \int_{g \in O(d)} g^{\otimes n} dg \in \mathbb{M}_d(\mathbb{C})^{\otimes n}$ is the orthogonal projection. Its image is exactly $\text{span}\{\rho(\mathfrak{m}), \mathfrak{m} \in \mathcal{M}(2n)\}$.*

Proof. First, observe that due to the invariance of the Haar measure,

$$P^2 = \int_{g_1, g_2 \in O(d)} g_1^{\otimes n} g_2^{\otimes n} dg_1 dg_2 = \int_{g_1, g_2 \in O(d)} (g_1 g_2)^{\otimes n} dg_1 dg_2 = \int_{g \in O(d)} g^{\otimes n} dg = P$$

Similarly,

$$P^* = \int_{g^* \in O(d)} g^{\otimes n} dg = \int_{g^{-1} \in O(d)} g^{\otimes n} dg = \int_{g \in O(d)} g^{\otimes n} dg = P.$$

So, P is an orthogonal projection. If n is odd, the $1 \rightarrow -1$ symmetry tells that the above integral is 0, therefore P is the zero projection. In the even case, by definition and due to proposition 2.2, an element $x \in V^{\otimes 2n}$ satisfies $g^{\otimes 2n} \cdot x = x$ iff $x \in \text{span}\{\rho(\mathfrak{m}), \mathfrak{m} \in \mathcal{M}(2n)\}$. This proves that the image of P is as claimed in the statement of the lemma. \square

The following lemma could stand as an euclidean geometry course exercise, and explains how to compute the coordinates of a projection onto the span of a set.

Proposition 2.4. *In an euclidean (or Hermitian) space with scalar (resp. Hermitian) product $\langle \cdot, \cdot \rangle$, let v be an element and P be an orthogonal projection, and (v_1, \dots, v_l) be a generating family of the image of this orthogonal projection.*

Let $Gr \in \mathbb{M}_l(\mathbb{C})$ be the Gram matrix associated with this basis, namely $Gr_{ij} = \langle v_i, v_j \rangle$, $X = (x_i)$ be the column vector given by $x_i = \langle v, v_i \rangle$, and W be a symmetric real matrix satisfying $Gr \cdot W \cdot Gr = Gr$.

There exists a vector $Y = (y_i)$ be a column vector such that $P(v) = \sum y_i v_i$. Besides this vector can be chosen to satisfy

$$Y = W \cdot X.$$

Note that we do not require the family $\{v_i\}$ to be independent in the hypothesis.

Proof. Although we will eventually need our proof in an Hermitian setting, let us prove it in the Euclidean setting in order to simplify the notation. The proof in the Hermitian setting goes along the same lines with the obvious modifications.

First, the existence of Y is clear, as it is equivalent to the existence of an orthogonal projection. Now, let us observe that

$$X = Gr \cdot Y.$$

Indeed this is the matrix notation for the fact that $P(v)$ is characterized by

$$\langle P(v), v_i \rangle = \langle v, v_i \rangle$$

for all i , and this characterizes $P(v)$ uniquely together with the fact that it is in $\text{span}\{v_i\}$. Therefore $X \in \text{Im}(Gr)$. Now, setting Y as $Y := W \cdot X$ makes it an acceptable choice for Y , in the sense that $X = Gr \cdot Y$. Indeed, this follows from the fact that $Gr \cdot W \cdot Gr = Gr$ and that X is in the image of Gr . So this completes the proof. \square

Now we are ready to proceed to the proof of Theorem 2.1:

Proof of Theorem 2.1: By the definition of P in Lemma 2.3, the canonical matrix element $P_{i_1, \dots, i_{2n}; j_1, \dots, j_{2n}}$ is exactly $\int_{g \in O(d)} g_{i_1 j_1} \cdots g_{i_{2n} j_{2n}} dg$. Equivalently stated,

$$\int_{g \in O(d)} g_{i_1 j_1} \cdots g_{i_{2n} j_{2n}} dg = \langle P(e_{i_1} \otimes \dots \otimes e_{i_{2n}}, e_{j_1} \otimes \dots \otimes e_{j_{2n}}) \rangle$$

Let us first compute $P(e_{i_1} \otimes \dots \otimes e_{i_{2n}})$. According to Lemma 2.4, we have

$$\sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)} \rho(\mathbf{m}) \cdot \langle \rho(\mathbf{n}), e_{i_1} \otimes \dots \otimes e_{i_{2n}} \rangle \text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n}).$$

Taking the scalar product of the above vector equality against the vector $e_{j_1} \otimes \dots \otimes e_{j_{2n}}$ gives

$$\int_{g \in O(d)} g_{i_1 j_1} \cdots g_{i_{2n} j_{2n}} dg = \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)} \text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n}) \prod_{k=1}^n \delta_{i_{\mathbf{m}(2k-1)}, i_{\mathbf{m}(2k)}} \delta_{j_{\mathbf{n}(2k-1)}, j_{\mathbf{n}(2k)}},$$

as we can observe that for \mathbf{m} in $\mathcal{M}(2n)$, we have

$$\langle e_{i_1} \otimes \dots \otimes e_{i_{2n}}, \rho(\mathbf{m}) \rangle = \prod_{k=1}^n \delta_{i_{\mathbf{m}(2k-1)}, i_{\mathbf{m}(2k)}}$$

\square

We would like to draw the attention of the reader of the fact that for the formula to holds, it is not necessary to assume that $\text{Wg}_n^{O(d)}$ is the pseudo-inverse of $G_n^{O(d)}$. Actually, $\text{Wg}_n^{O(d)}$ could be replaced by any matrix satisfying $G_n^{O(d)} \text{Wg}_n^{O(d)} = G_n^{O(d)}$ without modifying the integration formula. However, the pseudo-inverse is a canonical choice, and it will turn out to be a convenient one in the sequel of the paper.

Let us finish this section by recalling the Weingarten integration formula in the unitary case.

Let $\mathcal{M}(2n)^U$ be the set of pair partitions on $\{1, 2, \dots, 2n\}$ pairing an element in $\{1, \dots, n\}$ with an element in $\{n+1, \dots, 2n\}$. This is a subset of $\mathcal{M}(2n)$. For two pair partitions in $\mathcal{M}(2n)^U$ as in the case of $\mathcal{M}(2n)$. Similarly, we define the unitary gram matrix $G_n^{U(d)} = (G^{O(d)}(\mathbf{m}, \mathbf{n}))_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)^U}$ by

$$G^{U(d)}(\mathbf{m}, \mathbf{n}) = d^{\text{loop}(\mathbf{m}, \mathbf{n})},$$

and the unitary Weingarten matrix $\text{Wg}_n^{U(d)}$ as the pseudo-inverse of $G_n^{U(d)}$. The following Theorem describes the unitary integral case, and it was proved in [CSn]:

Theorem 2.5 ([CSn]). *Given indices i_1, \dots, i_{2n} and j_1, \dots, j_{2n} in $\{1, 2, \dots, d\}$,*

$$(2.3) \quad \int_{g \in U(d)} g_{i_1 j_1} \cdots g_{i_n j_n} \overline{g_{i_{n+1} j_{n+1}}} \cdots \overline{g_{i_{2n} j_{2n}}} dg = \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)^U} \text{Wg}^{U(d)}(\mathbf{m}, \mathbf{n}) \prod_{k=1}^n \delta_{i_{\mathbf{m}(2k-1)}, i_{\mathbf{m}(2k)}} \delta_{j_{\mathbf{n}(2k-1)}, j_{\mathbf{n}(2k)}}.$$

Note that this statement is very similar to the orthogonal case. The main difference is that we have to consider a submatrix of the original Gram matrix $G^{O(d)}$. A proof similar to the orthogonal case could be carried along the same lines after adapting the invariant theory from the orthogonal case to the unitary case.

3 A new expression for the orthogonal Weingarten function

To each partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we call $\ell(\lambda)$ its *length* and $|\lambda|$ its *weight*. If $|\lambda| = n$, we write $\lambda \vdash n$. Let H_n be the centralizer of the element $(1, 2)(3, 4) \cdots (2n-1, 2n)$ in S_{2n} , where (i, j) is a transposition. The group H_n is called the *hyperoctahedral group*. It is of order $2^n n!$ and isomorphic to the wreath product $S_2 \wr S_n$. The pair (S_{2n}, H_n) is a *Gelfand pair*. Namely, the induced representation $\text{ind}_{H_n}^{S_{2n}}(1_{H_n})$ of the trivial representation 1_{H_n} of H_n is a direct sum of inequivalent irreducible representations of S_{2n} .

Following [Mac, VII.1 and 2], let us review a few elements of the theory of the Gelfand pair (S_{2n}, H_n) . If each pair partition \mathbf{m} in $\mathcal{M}(2n)$ is expressed by the form (2.1) with $\mathbf{m}(2i-1) < \mathbf{m}(2i)$ for $1 \leq i \leq n$ and with $\mathbf{m}(1) < \mathbf{m}(3) < \cdots < \mathbf{m}(2n-1)$, then we embed the set $\mathcal{M}(2n)$ into S_{2n} via the mapping

$$(3.1) \quad \mathcal{M}(2n) \ni \mathbf{m} \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2n \\ \mathbf{m}(1) & \mathbf{m}(2) & \mathbf{m}(3) & \mathbf{m}(4) & \cdots & \mathbf{m}(2n) \end{pmatrix} \in S_{2n}.$$

Pair partitions $\mathbf{m} \in \mathcal{M}(2n)$ are representatives of the left cosets σH_n of H_n in S_{2n} :

$$(3.2) \quad S_{2n} = \bigsqcup_{\mathbf{m} \in \mathcal{M}(2n)} \mathbf{m} H_n.$$

Consider the double cosets $H_n \sigma H_n$ in S_{2n} . These cosets are indexed by partitions of n :

$$(3.3) \quad S_{2n} = \bigsqcup_{\rho \vdash n} H_\rho.$$

Each double coset H_ρ is of cardinality $|H_\rho| = \frac{(2^n n!)^2}{z_{2\rho}}$. Here, for a partition μ ,

$$z_\mu = \prod_{i \geq 1} i^{m_i(\mu)} m_i(\mu)!$$

with the multiplicity $m_i(\mu)$ of i in μ . The permutation $\sigma \in S_{2n}$ is said to be of *coset-type* ρ and written as $\Xi(\sigma) = \rho$ if $\sigma \in H_\rho$.

The partition $\Xi(\sigma)$ is also defined as follows. Consider the graph $\Gamma(\sigma)$ whose vertex set is $\{1, 2, \dots, 2n\}$ and whose edge set consists of $\{2i-1, 2i\}$ and $\{\sigma(2i-1), \sigma(2i)\}$, $1 \leq i \leq n$. Then

$\Gamma(\sigma)$ has some connected components of even lengths $2\rho_1 \geq 2\rho_2 \geq \dots$, say. Thus σ determines a partition $\rho = (\rho_1, \rho_2, \dots)$ of n . This ρ is the coset-type $\Xi(\sigma)$. Two permutations $\sigma_1, \sigma_2 \in S_{2n}$ have the same coset-type if and only if $\sigma_1 \in H_n \sigma_2 H_n$.

For each partition λ of n , we define the *zonal spherical functions* of the Gelfand pair (S_{2n}, H_n) by

$$(3.4) \quad \omega^\lambda(\sigma) = \frac{1}{2^n n!} \sum_{\zeta \in H_n} \chi^{2\lambda}(\sigma\zeta), \quad \sigma \in S_{2n},$$

where $\chi^{2\lambda}$ is the irreducible character of S_{2n} associated with $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$. These functions are constant on each double coset H_ρ . We denote by ω_ρ^λ the value of ω^λ on the double coset H_ρ . Put $f^{2\lambda} = \chi^{2\lambda}(\text{id}_n)$, where id_n is the identity permutation in S_n .

For each $n \geq 1$ and each partition $\lambda \vdash n$, the *zonal polynomial* Z_λ is a symmetric polynomial defined by

$$(3.5) \quad Z_\lambda = 2^n n! \sum_{\rho \vdash n} z_{2\rho}^{-1} \omega_\rho^\lambda p_\rho,$$

where $p_k(x_1, x_2, \dots) = x_1^k + x_2^k + \dots$ is the power-sum symmetric polynomial and $p_\rho = p_{\rho_1} p_{\rho_2} \dots$. The zonal polynomial Z_λ is the Jack symmetric polynomial $J_\lambda^{(\alpha)}$ with parameter $\alpha = 2$. The specialization $Z_\lambda(1^d) = Z_\lambda(1, \dots, 1)$ is given by

$$(3.6) \quad Z_\lambda(1^d) = \prod_{(i,j) \in \lambda} (d + 2j - i - 1),$$

where (i, j) run over all squares of the Young diagram of λ , i.e., $i = 1, 2, \dots, \ell(\lambda)$ and $j = 1, 2, \dots, \lambda_i$. In particular, $Z_\lambda(1^d)$ is zero if and only if $\ell(\lambda) > d$.

Now, entries of the Gram matrix are expressed by $G^{O(d)}(\mathbf{m}, \mathbf{n}) = d^{\ell(\Xi(\mathbf{m}^{-1}\mathbf{n}))}$. Here \mathbf{m}, \mathbf{n} are regarded as permutations in S_{2n} . The following Theorem is our main result.

Theorem 3.1. *Let d, n be positive integers. For $\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)$, we have*

$$(3.7) \quad \text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n}) = \frac{2^n n!}{(2n)!} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} f^{2\lambda} \frac{\omega^\lambda(\mathbf{m}^{-1}\mathbf{n})}{Z_\lambda(1^d)}.$$

Before proving this Theorem, in the forthcoming section 4, let us illustrate it by an example:

Example 3.1. Let $n = 2$. We have $\omega^{(2)}(\mathbf{m}^{-1}\mathbf{n}) = 1$ for any $\mathbf{m}, \mathbf{n} \in \mathcal{M}(4)$, and $\omega^{(1,1)}(\mathbf{m}^{-1}\mathbf{n}) = 1$ if $\mathbf{m} = \mathbf{n}$, and $\omega^{(1,1)}(\mathbf{m}^{-1}\mathbf{n}) = -\frac{1}{2}$ if $\mathbf{m} \neq \mathbf{n}$. Therefore if $\mathbf{m} = \mathbf{n}$ and $d \geq 2$, we have

$$\text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n}) = \frac{1}{3} \left(\frac{1}{d(d+2)} + \frac{2}{d(d-1)} \right) = \frac{d+1}{d(d+2)(d-1)}.$$

If $\mathbf{m} \neq \mathbf{n}$ and $d \geq 2$, we have

$$\text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n}) = \frac{1}{3} \left(\frac{1}{d(d+2)} + \frac{2 \cdot (-\frac{1}{2})}{d(d-1)} \right) = \frac{-1}{d(d+2)(d-1)}.$$

If $d = 1$, then $\text{Wg}^{O(1)}(\mathbf{m}, \mathbf{n}) = \frac{1}{9}$ for any $\mathbf{m}, \mathbf{n} \in \mathcal{M}(4)$. In general, for any $\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)$ and $d = 1$, we have

$$\text{Wg}^{O(1)}(\mathbf{m}, \mathbf{n}) = \frac{2^n n!}{(2n)!} f^{(2n)} \frac{\omega^{(n)}(\mathbf{m}^{-1} \mathbf{n})}{Z_{(n)}(1^1)} = \left(\frac{2^n n!}{(2n)!} \right)^2.$$

To finish this section, observe that a result similar to Theorem 3.1 has already been obtained in [CSn]. We identify the set $\mathcal{M}(2n)^U$ with the permutations on n points by associating to $\mathbf{m} \in \mathcal{M}(2n)^U$ the permutation σ satisfying $\sigma(i) = j$ iff \mathbf{m} links i and $n + j$. According to this identification, we can multiply elements of $\mathcal{M}(2n)^U$ according to the symmetric group structure, or consider their inverse. We can also consider their conjugacy classes. With the above notation and notations parallel to our main Theorem 3.1, we have

Theorem 3.2. *Let d, n be positive integers. For $\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)^U$, we have*

$$(3.8) \quad \text{Wg}^{U(d)}(\mathbf{m}, \mathbf{n}) = \frac{1}{n!^2} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} \frac{(f^\lambda)^2 \chi^\lambda(\mathbf{m}^{-1} \mathbf{n})}{s_\lambda(1^d)},$$

where s_λ is the Schur polynomial and χ^λ is the non-normalized character of the symmetric group.

Although the proof of the above Theorem 3.2 was not so involved, the proof of the main Theorem 3.1 requires less standard algebraic material such as Jack polynomial theory (and in particular, zonal polynomials), and Gelfand pair theory. The next section is devoted to the review of this material and the proof of Theorem 3.1.

4 Proof of Theorem 3.1

Our strategy of proof consists in representing $G_n^{O(d)}$ as an H_n bi-invariant function on the symmetric group S_{2n} , and performing a Fourier transform on it. Consider the \mathbb{C} -vector space

$$\mathbb{C}[\mathcal{M}(2n)] = \bigoplus_{\mathbf{m} \in \mathcal{M}(2n)} \mathbb{C} \mathbf{m}$$

and the representation ρ_{2n} of S_{2n} on $\mathbb{C}[\mathcal{M}(2n)]$ defined by

$$(4.1) \quad \rho_{2n}(\sigma) \mathbf{m} = \{ \{ \sigma(\mathbf{m}(1)), \sigma(\mathbf{m}(2)) \}, \dots, \{ \sigma(\mathbf{m}(2n-1)), \sigma(\mathbf{m}(2n)) \} \}.$$

The pair partition \mathbf{m} and $\rho_{2n}(\sigma) \mathbf{m}$ are identified with elements of S_{2n} via (3.1). However, the permutation associated with $\rho_{2n}(\sigma) \mathbf{m}$ may be different from the product $\sigma \mathbf{m}$ in S_{2n} . By the left coset decomposition (3.2), there exists a unique permutation $\zeta_{\sigma, \mathbf{m}} \in H_n$ such that

$$(4.2) \quad \rho_{2n}(\sigma) \mathbf{m} = \sigma \mathbf{m} \zeta_{\sigma, \mathbf{m}} \in S_{2n}.$$

Example 4.1. Let $\sigma = (1, 3, 2)(4) \in S_4$ and $\mathbf{m} = \{ \{1, 3\}, \{2, 4\} \} \in \mathcal{M}(4)$. Then we have

$$\rho_4(\sigma) \mathbf{m} = \{ \{ \sigma(1), \sigma(3) \}, \{ \sigma(2), \sigma(4) \} \} = \{ \{3, 2\}, \{1, 4\} \} = \{ \{1, 4\}, \{2, 3\} \},$$

and so $\rho_4(\sigma) \mathbf{m}$ is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = (2, 4, 3)$ as an element in S_4 , which equals $\sigma \mathbf{m} \zeta_{\sigma, \mathbf{m}}$ with $\zeta_{\sigma, \mathbf{m}} = (1, 2)(1, 3)(2, 4) \in H_2$.

Any matrix $L = (L(\mathbf{m}, \mathbf{n}))_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)}$ is identified with the endomorphism T_L on $\mathbb{C}[\mathcal{M}(2n)]$ defined by

$$(4.3) \quad T_L(\mathbf{m}) = \sum_{\mathbf{n} \in \mathcal{M}(2n)} L(\mathbf{m}, \mathbf{n}) \mathbf{n}.$$

From now, we abbreviate the Gram matrix $G_n^{O(d)}$ and Weingarten matrix $Wg_n^{O(d)}$ to G and Wg , respectively.

Lemma 4.1. *For the Gram matrix G , we have $T_G(\rho_{2n}(\sigma)\mathbf{m}) = \rho_{2n}(\sigma)T_G(\mathbf{m})$ for all $\sigma \in S_{2n}$ and $\mathbf{m} \in \mathcal{M}(2n)$. Hence $T_G \in \text{End}_{\rho_{2n}(\mathbb{C}[S_{2n}])}(\mathbb{C}[\mathcal{M}(2n)])$.*

Proof. For each $\sigma \in S_{2n}$ and $\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)$, we have

$$G(\rho_{2n}(\sigma)\mathbf{m}, \rho_{2n}(\sigma)\mathbf{n}) = d^{\ell(\Xi((\sigma\mathbf{m}\zeta_{\sigma,\mathbf{m}})^{-1}(\sigma\mathbf{n}\zeta_{\sigma,\mathbf{n}})))} = d^{\ell(\Xi(\zeta_{\sigma,\mathbf{m}}^{-1}\mathbf{m}^{-1}\mathbf{n}\zeta_{\sigma,\mathbf{n}}))} = d^{\ell(\Xi(\mathbf{m}^{-1}\mathbf{n}))} = G(\mathbf{m}, \mathbf{n})$$

because $\Xi(\zeta\sigma\zeta') = \Xi(\sigma)$ for all $\sigma \in S_{2n}$ and $\zeta, \zeta' \in H_n$. Hence

$$\begin{aligned} T_G(\rho_{2n}(\sigma)\mathbf{m}) &= \sum_{\mathbf{n} \in \mathcal{M}(2n)} G(\rho_{2n}(\sigma)\mathbf{m}, \mathbf{n}) \mathbf{n} = \sum_{\mathbf{n} \in \mathcal{M}(2n)} G(\rho_{2n}(\sigma)\mathbf{m}, \rho_{2n}(\sigma)\mathbf{n}) \rho_{2n}(\sigma)\mathbf{n} \\ &= \sum_{\mathbf{n} \in \mathcal{M}(2n)} G(\mathbf{m}, \mathbf{n}) \rho_{2n}(\sigma)\mathbf{n} = \rho_{2n}(\sigma)T_G(\mathbf{m}). \end{aligned}$$

□

Let

$$e = \frac{1}{2^n n!} \sum_{\zeta \in H_n} \zeta.$$

This element satisfies $e^2 = e$. The set $\{\mathbf{m}e \mid \mathbf{m} \in \mathcal{M}(2n)\}$ is a basis of the left $\mathbb{C}[S_{2n}]$ -module $\mathbb{C}[S_{2n}]e$. The proof of the following lemma can be found, for example, in [Mac, VII.1]:

Lemma 4.2. *The endomorphism ring $\text{End}_{\mathbb{C}[S_{2n}]}(\mathbb{C}[S_{2n}]e)$ is anti-isomorphic to $e\mathbb{C}[S_{2n}]e$, the anti-isomorphism being $\phi \mapsto \phi(e)$.*

By definition, zonal spherical functions ω^λ for the Gelfand pair (S_{2n}, H_n) are expressed as $\omega^\lambda = \chi^{2\lambda}e$.

Lemma 4.3. *Zonal spherical functions ω^λ satisfy the following properties.*

1. $\omega^\lambda \omega^\mu = \delta_{\lambda\mu} \frac{(2n)!}{f^{2\lambda}} \omega^\lambda$.
2. $\{\omega^\lambda \mid \lambda \vdash n\}$ is a basis of $e\mathbb{C}[S_{2n}]e$.
3. Let $\mu = (\mu_1, \mu_2, \dots) \vdash n$, and let $p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\ell(\mu)}$ be the power-sum symmetric polynomial. Then we have

$$(4.4) \quad p_\mu(x_1, x_2, \dots, x_r) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} f^{2\lambda} \omega_\mu^\lambda Z_\lambda(x_1, \dots, x_r).$$

Here x_1, \dots, x_r are indeterminates and r is a positive integer.

Proof. The first and second claims are seen in [Mac, VII (1.4)]. The third claim is seen in [Mac, VII (2.16)]. \square

Let \mathcal{I} be the bijective linear map from $\mathbb{C}[\mathcal{M}(2n)]$ to $\mathbb{C}[S_{2n}]e$ defined by $\mathcal{I}(\mathbf{m}) = \mathbf{m}e$ ($\mathbf{m} \in \mathcal{M}(2n)$).

Lemma 4.4. *The map \mathcal{I} is an intertwiner. Hence $\mathbb{C}[\mathcal{M}(2n)]$ with the action ρ_{2n} is isomorphic to $\mathbb{C}[S_{2n}]e$ as $\mathbb{C}[S_{2n}]$ -modules.*

Proof. For each $\sigma \in S_{2n}$ and $\mathbf{m} \in \mathcal{M}(2n)$, we have

$$\mathcal{I}(\rho_{2n}(\sigma)\mathbf{m}) = (\rho(\sigma)\mathbf{m})e = (\sigma\mathbf{m}\zeta_{\sigma,\mathbf{m}})e = \sigma(\mathbf{m}\zeta_{\sigma,\mathbf{m}}e) = \sigma(\mathbf{m}e) = \sigma\mathcal{I}(\mathbf{m})$$

since $\zeta e = e$ for all $\zeta \in H_n$. \square

By Lemma 4.1 and Lemma 4.4, the endomorphism $\mathcal{I} \circ T_G \circ \mathcal{I}^{-1}$ belongs to $\text{End}_{\mathbb{C}[S_{2n}]}(\mathbb{C}[S_{2n}]e)$. Moreover, it follows from Lemma 4.2 that $\hat{G} := (\mathcal{I} \circ T_G \circ \mathcal{I}^{-1})(e)$ belongs to $e\mathbb{C}[S_{2n}]e$ and

$$\hat{G} = \sum_{\mathbf{m} \in \mathcal{M}(2n)} d^{\ell(\Xi(\mathbf{m}))} \mathbf{m}e.$$

If we specialize as $(x_1, x_2, \dots) = (1^d)$ in (4.4), we obtain

$$(4.5) \quad d^{\ell(\mu)} = \frac{2^n n!}{(2n)!} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} f^{2\lambda} \omega_\mu^\lambda Z_\lambda(1^d).$$

Therefore we obtain

$$\hat{G} = \frac{2^n n!}{(2n)!} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} f^{2\lambda} Z_\lambda(1^d) \sum_{\mathbf{m} \in \mathcal{M}(2n)} \omega_{\Xi(\mathbf{m})}^\lambda \mathbf{m}e.$$

Since, by the left coset decomposition (3.2),

$$2^n n! \sum_{\mathbf{m} \in \mathcal{M}(2n)} \omega_{\Xi(\mathbf{m})}^\lambda \mathbf{m}e = \sum_{\mathbf{m} \in \mathcal{M}(2n)} \sum_{\zeta \in H_n} \omega^\lambda(\mathbf{m}\zeta) \mathbf{m}\zeta = \sum_{\sigma \in S_{2n}} \omega^\lambda(\sigma) \sigma = \omega^\lambda,$$

we can express as

$$(4.6) \quad \hat{G} = \frac{1}{(2n)!} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} f^{2\lambda} Z_\lambda(1^d) \omega^\lambda.$$

Define the element $\hat{W}g$ in $e\mathbb{C}[S_{2n}]e$ by

$$(4.7) \quad \hat{W}g = \frac{1}{(2n)!} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} f^{2\lambda} Z_\lambda(1^d)^{-1} \omega^\lambda.$$

By Claim 1 of Lemma 4.3 we obtain the following proposition.

Proposition 4.5. *It holds that*

$$\hat{G} \cdot \hat{W}g = \frac{1}{(2n)!} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} f^{2\lambda} \omega^\lambda.$$

Moreover, we have

$$\hat{G} \cdot \hat{W}g \cdot \hat{G} = \hat{G}, \quad \hat{W}g \cdot \hat{G} \cdot \hat{W}g = \hat{W}g.$$

As the matrix G is corresponding to \hat{G} , the matrix Wg given in (3.7) is corresponding to $\hat{W}g$. Hence, since G is symmetric, Wg is the pseudo-inverse matrix of G by Proposition 4.5.

Using Claim 1 and 2 of Lemma 4.3, we obtain

$$(4.8) \quad e = \frac{1}{(2n)!} \sum_{\lambda \vdash n} f^{2\lambda} \omega^\lambda$$

because $e^2 = e$. The element e is the identity of the algebra $e\mathbb{C}[S_{2n}]e$ and $\hat{G} \cdot \hat{W}g = e$ if $d \geq n$. Hence, when $d \geq n$, the matrix G is invertible and $Wg = G^{-1}$.

5 Moments of the trace of a truncated orthogonal matrix

As an application of Theorem 2.1 and 3.1, we calculate the moment of the trace of a truncated orthogonal random matrix.

Theorem 5.1. *Let k, d, n be positive integers and suppose $k \leq d$. Let $g = (g_{ij})_{1 \leq i, j \leq d}$ be a random matrix chosen with respect to the Haar measure from the orthogonal group $O(d)$, and let $g^{(k)}$ be its $k \times k$ upper left corner: $g^{(k)} = (g_{ij})_{1 \leq i, j \leq k}$. Then we have*

$$(5.1) \quad \int_{g \in O(d)} (\text{tr}(g^{(k)}))^{2n} dg = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq k}} f^{2\lambda} \frac{Z_\lambda(1^k)}{Z_\lambda(1^d)}.$$

We recover Rains's result in [R] by setting $k = d$ in Theorem 5.1. The average of the odd power of $\text{tr}(g^{(k)})$ is zero:

$$(5.2) \quad \int_{g \in O(d)} (\text{tr}(g^{(k)}))^{2n-1} dg = 0.$$

The unitary group version for Theorem 5.1 is seen in [N, Theorem 1].

Proof. A straightforward calculation and Theorem 2.1 give

$$\begin{aligned}
& \int_{g \in O(d)} (\text{tr}(g^{(k)}))^{2n} dg = \sum_{a_1=1}^k \cdots \sum_{a_{2n}=1}^k \int_{g \in O(d)} g_{a_1 a_1} \cdots g_{a_{2n} a_{2n}} dg \\
&= \sum_{a_1=1}^k \cdots \sum_{a_{2n}=1}^k \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)} \text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n}) \prod_{p=1}^n \delta_{a_{\mathbf{m}(2p-1)}, a_{\mathbf{m}(2p)}} \delta_{a_{\mathbf{n}(2p-1)}, a_{\mathbf{n}(2p)}} \\
&= \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)} \text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n}) \sum_{a_1=1}^k \cdots \sum_{a_{2n}=1}^k \prod_{p=1}^n \delta_{a_{\mathbf{m}(2p-1)}, a_{\mathbf{m}(2p)}} \delta_{a_{\mathbf{n}(2p-1)}, a_{\mathbf{n}(2p)}} \\
(5.3) \quad &= \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)} \text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n}) k^{\text{loop}(\mathbf{m}, \mathbf{n})}.
\end{aligned}$$

This coincides with the trace of the matrix $\text{Wg}_n^{O(d)} \cdot G_n^{O(k)}$.

By (4.5), we have

$$k^{\text{loop}(\mathbf{m}, \mathbf{n})} = k^{\ell(\Xi(\mathbf{m}^{-1}\mathbf{n}))} = \frac{2^n n!}{(2n)!} \sum_{\substack{\mu \vdash n \\ \ell(\mu) \leq k}} f^{2\mu} Z_\mu(1^k) \omega^\mu(\mathbf{m}^{-1}\mathbf{n}).$$

It follows from this equation and Theorem 3.1 that (5.3) equals

$$\left(\frac{2^n n!}{(2n)!} \right)^2 \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} \sum_{\substack{\mu \vdash n \\ \ell(\mu) \leq k}} f^{2\lambda} f^{2\mu} \frac{Z_\mu(1^k)}{Z_\lambda(1^d)} \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)} \omega^\lambda(\mathbf{m}^{-1}\mathbf{n}) \omega^\mu(\mathbf{m}^{-1}\mathbf{n}).$$

Now it is enough to prove the following identity: for any $\lambda, \mu \vdash n$,

$$(5.4) \quad \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)} \omega^\lambda(\mathbf{m}^{-1}\mathbf{n}) \omega^\mu(\mathbf{m}^{-1}\mathbf{n}) = \left(\frac{(2n)!}{2^n n!} \right)^2 \frac{1}{f^{2\lambda}} \delta_{\lambda\mu}.$$

By the coset decomposition in (3.2), we see that

$$\begin{aligned}
& \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)} \omega^\lambda(\mathbf{m}^{-1}\mathbf{n}) \omega^\mu(\mathbf{m}^{-1}\mathbf{n}) \\
&= \left(\frac{1}{2^n n!} \right)^2 \sum_{\zeta, \zeta' \in H_n} \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)} \omega^\lambda((\mathbf{m}\zeta)^{-1}(\mathbf{n}\zeta')) \omega^\mu((\mathbf{m}\zeta)^{-1}(\mathbf{n}\zeta')) \\
&= \left(\frac{1}{2^n n!} \right)^2 \sum_{\sigma, \tau \in S_{2n}} \omega^\lambda(\sigma^{-1}\tau) \omega^\mu(\sigma^{-1}\tau) \\
&= \frac{(2n)!}{(2^n n!)^2} \sum_{\sigma \in S_{2n}} \omega^\lambda(\sigma) \omega^\mu(\sigma) = \left(\frac{(2n)!}{2^n n!} \right)^2 \langle \omega^\lambda, \omega^\mu \rangle_{S_{2n}}.
\end{aligned}$$

Here $\langle \cdot, \cdot \rangle_{S_{2n}}$ is the inner product on $\mathbb{C}[S_{2n}]$. Since $\langle \omega^\lambda, \omega^\mu \rangle_{S_{2n}} = \frac{1}{f^{2\lambda}} \delta_{\lambda\mu}$ (see [Mac, VII (1.4) (iv)]), the equality (5.4) holds true. \square

Observe that in the first computation of the above proof also lead in the earlier paper [BCo] to the notion of truncated characters for compact quantum groups.

6 The full cycle case

In this section, we suppose $d \geq n$. We are interested in $\text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n})$ where the coset type of $\mathbf{m}^{-1}\mathbf{n}$ is the 1-length partition (n) . The motivation for this comes from the paper [Co]. Indeed, in the unitary case, the following was proved:

Proposition 6.1. *Given a partition μ of n , let $\text{Wg}^U(\mu, d) := \text{Wg}^{U(d)}(\mathbf{m}, \mathbf{n})$, where $\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)^U$ and the coset type of $\mathbf{m}^{-1}\mathbf{n}$ is μ . (Recall (3.8)). Then the following holds true:*

$$(6.1) \quad \text{Wg}^U((n), d) = (-1)^{n-1} c_{n-1} \prod_{-n+1 \leq j \leq n-1} (d-j)^{-1}$$

Here $c_k = \frac{(2k)!}{(k+1)!k!}$ is the Catalan number.

Sketch of proof. Indeed, a classical combinatorial result about the Schur polynomials shows that if one writes $\text{Wg}^U((n), d)$ as an irreducible rational fraction, its denominator has to be $\prod_{-q+1 \leq j \leq q-1} (d-j)$. An explicit formula for the asymptotic value as $d \rightarrow \infty$ of $\text{Wg}^U((n), d)$ given in [Co], together with the fact that $\text{Wg}^U((n), d)$ has to be a rational fraction concludes the proof. \square

Note that we found it conceptually strange that an asymptotic estimate was necessary to prove such an exact algebraic formula. One can actually observe that the knowledge of asymptotics can actually be bypassed by an application of the Zeilberger algorithms implemented on many formal algebraic computing softwares (e.g. maple or mathematica). See [PWZ] for details. Related generalizations of this results are in preparation in a paper by the second author and Novak, [MN].

In the orthogonal case, the entries $\text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n})$ of the Weingarten matrix only depend on the coset type $\Xi(\mathbf{m}^{-1}\mathbf{n})$. For each partition μ of n and a positive integer $d \geq n$, we put

$$(6.2) \quad \text{Wg}^O(\mu, d) = \frac{1}{(2n-1)!!} \sum_{\lambda \vdash n} \frac{f^{2\lambda} \omega_\mu^\lambda}{Z_\lambda(1^d)}.$$

Theorem 3.1 implies that if $\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)$ and $\mu = \Xi(\mathbf{m}^{-1}\mathbf{n})$ then $\text{Wg}^O(\mu, d) = \text{Wg}^{O(d)}(\mathbf{m}, \mathbf{n})$.

Consider $\text{Wg}^O((n), d)$. We know values $\omega_{(n)}^\lambda$ (e.g. [Mac, VII.2, Example 2(c)]):

$$\omega_{(n)}^\lambda = \frac{1}{2^{n-1}(n-1)!} \prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (1,1)}} (2j-i-1).$$

In particular, $\omega_{(n)}^\lambda = 0$ if the square $(3, 2)$ is contained in the Young diagram of λ , i.e., $\lambda_3 \geq 2$. Therefore we obtain an expression

$$(6.3) \quad \text{Wg}^O((n), d) = \frac{1}{(2n-1)!} \sum_{\substack{\lambda \vdash n \\ \lambda_3 \leq 1}} \frac{f^{2\lambda}}{d} \prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (1,1)}} \frac{2j-i-1}{d+2j-i-1}.$$

If we reduce to a common denominator, the denominator has the form $\prod_{(i,j)}(d+2j-i-1)$, where (i,j) run over the set

$$\begin{aligned} & \{(i,j) \mid \text{there exists } \lambda \vdash n \text{ such that } \lambda_3 \leq 1 \text{ and } (i,j) \in \lambda\} \\ & = \{(1,j) \mid j = 1, 2, \dots, n\} \sqcup \{(i,1) \mid i = 2, 3, \dots, n\} \sqcup \{(2,j) \mid j = 2, 3, \dots, \lfloor \frac{n}{2} \rfloor\}. \end{aligned}$$

Now we define the polynomial $P_n(d)$ in d with \mathbb{Q} -coefficients via

$$(6.4) \quad \text{Wg}^O((n), d) = \frac{(-1)^{n-1} P_n(d)}{d \prod_{j=1}^{n-1} (d+2j)(d-j) \cdot \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} (d+2k-1)}.$$

Proposition 6.2. *The polynomials $P_n(d)$ satisfy the following properties.*

- (i) *For $n \geq 2$, the degree of P_n is exactly $\lfloor \frac{n}{2} \rfloor - 1$.*
- (ii) *The coefficient of the highest-degree term of P_n equals the Catalan number c_{n-1} .*
- (iii) *The constant term of P_n equals*

$$P_n(0) = \sum_{\substack{\lambda \vdash n \\ \lambda_3 \leq 1}} f^{2\lambda} \times \frac{(n-1)!}{(2n-1)!!} \times \begin{cases} (n-4)!! & \text{if } n \text{ is odd and } n \geq 5 \\ (n-3)!! & \text{if } n \text{ is even and } n \geq 4 \\ 1 & \text{otherwise.} \end{cases}$$

Proof. We use a result for the asymptotics of the Weingarten function, obtained in [CSn]: for a partition μ of n , as $d \rightarrow \infty$,

$$\text{Wg}^O(\mu, d) = \left(\prod_{i \geq 1} (-1)^{\mu_i - 1} c_{\mu_i - 1} \right) d^{-2n + \ell(\mu)} (1 + O(d^{-1})).$$

In particular, $\text{Wg}^O((n), d) = (-1)^{n-1} c_{n-1} d^{-2n+1} (1 + O(d^{-1}))$. This fact and equation (6.4) imply our desired first and second claims. The third claim follows from (6.3) and (6.4) with $d \rightarrow 0$. \square

Suppose $\lambda \vdash n$ satisfies $\lambda_3 \leq 1$. Then we can write as $\lambda = (r, s, 1^{n-r-s})$ with $r = 1, 2, \dots, n$ and $s = 0, 1, \dots, \min\{r, n-r\}$. By the well-known hook formula for $f^{2\lambda}$, we obtain that if $s \geq 1$,

$$\begin{aligned} \text{hook}(r, s, n) &:= \frac{(2n)!}{f^{2\lambda}} = \frac{(2n)!}{f^{(2r, 2s, 2^{n-r-s})}} \\ &= (n+r-s+1)(n+r-s)(n-r+s)(n-r+s-1) \cdot (n-r-s+1)! \cdot (n-r-s)! \\ &\quad \cdot (2s-2)! \cdot \frac{(2r-1)!}{2r-2s+1}. \end{aligned}$$

Now by (6.3) and (6.4) we have

(6.5)

$$P_n(d) = (-1)^{n-1} 2n \prod_{j=1}^{n-1} (d+2j)(d-j) \cdot \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} (d+2k-1) \left(\frac{1}{(2n)!} \prod_{j=2}^n \frac{2j-2}{d+2j-2} \right. \\ \left. + \sum_{r=1}^{n-1} \sum_{s=1}^{\min(r, n-r)} \frac{1}{\text{hook}(r, s, n)} \prod_{j=2}^r \frac{2j-2}{d+2j-2} \cdot \prod_{j=1}^s \frac{2j-3}{d+2j-3} \cdot \prod_{i=3}^{n-r-s+2} \frac{-(i-1)}{d-(i-1)} \right).$$

For example,

$$P_1(d) = P_2(d) = 1. \quad P_3(d) = 2. \quad P_4(d) = 5d + 6. \quad P_5(d) = 14d + 24.$$

$$P_6(d) = 42d^2 + 236d + 344. \quad P_7(d) = 132d^2 + 920d + 1824.$$

$$P_8(d) = 429d^3 + 5924d^2 + 29116d + 51600.$$

$$P_9(d) = 1430d^3 + 23124d^2 + 138352d + 305280.$$

$$P_{10}(d) = 4862d^4 + 126816d^3 + 1326016d^2 + 6598896d + 13071744.$$

From those examples, we suggest the following conjecture.

Conjecture 6.3. *All of coefficients of $P_n(d)$ are nonnegative integers.*

7 Table of value

We finish this paper by giving a table of values of the orthogonal Weingarten functions up to $n = 6$. To do it, we employ the table of zonal polynomials [PJ]. Observe that even with $n = 4$, this is a quite heavy formal computation without our formula, and a standard formal algebra system on a modern desktop can not go beyond $n = 4$ (105×105 formal matrix to invert) without our formula provided in this paper. We expect that our formula should have quite interesting applications quite soon. The table for $n \leq 4$ is also seen in [CSn].

Table goes here:

$$\text{Wg}^O((1), d) = \frac{1}{d}.$$

$$\text{Wg}^O((2), d) = \frac{1}{3} \left(\frac{1 \cdot 1}{d(d+2)} + \frac{2 \cdot (-\frac{1}{2})}{d(d-1)} \right) = \frac{-1}{d(d+2)(d-1)}.$$

$$\text{Wg}^O((1^2), d) = \frac{1}{3} \left(\frac{1 \cdot 1}{d(d+2)} + \frac{2 \cdot 1}{d(d-1)} \right) = \frac{d+1}{d(d+2)(d-1)}.$$

$$\begin{aligned}
\text{Wg}^O((3), d) &= \frac{1}{15} \left(\frac{1 \cdot 1}{d(d+2)(d+4)} + \frac{9 \cdot (-\frac{1}{4})}{d(d+1)(d-1)} + \frac{5 \cdot \frac{1}{4}}{d(d-1)(d-2)} \right) \\
&= \frac{2}{d(d+2)(d+4)(d-1)(d-2)}. \\
\text{Wg}^O((2, 1), d) &= \frac{1}{15} \left(\frac{1 \cdot 1}{d(d+2)(d+4)} + \frac{9 \cdot \frac{1}{6}}{d(d+1)(d-1)} + \frac{5 \cdot (-\frac{1}{2})}{d(d-1)(d-2)} \right) \\
&= \frac{-1}{d(d+4)(d-1)(d-2)}. \\
\text{Wg}^O((1^3), d) &= \frac{1}{15} \left(\frac{1 \cdot 1}{d(d+2)(d+4)} + \frac{9 \cdot 1}{d(d+1)(d-1)} + \frac{5 \cdot 1}{d(d-1)(d-2)} \right) \\
&= \frac{d^2 + 3d - 2}{d(d+2)(d+4)(d-1)(d-2)}.
\end{aligned}$$

$$\begin{aligned}
\text{Wg}^O((4), d) &= \frac{-(5d+6)}{d(d+1)(d+2)(d+4)(d+6)(d-1)(d-2)(d-3)}. \\
\text{Wg}^O((3, 1), d) &= \frac{2}{(d+1)(d+2)(d+6)(d-1)(d-2)(d-3)}. \\
\text{Wg}^O((2, 2), d) &= \frac{d^2 + 5d + 18}{d(d+1)(d+2)(d+4)(d+6)(d-1)(d-2)(d-3)}. \\
\text{Wg}^O((2, 1, 1), d) &= \frac{-d^3 - 6d^2 - 3d + 6}{d(d+1)(d+2)(d+4)(d+6)(d-1)(d-2)(d-3)}. \\
\text{Wg}^O((1^4), d) &= \frac{(d+3)(d^2 + 6d + 1)}{d(d+1)(d+2)(d+4)(d+6)(d-1)(d-3)}.
\end{aligned}$$

$$\begin{aligned}
\text{Wg}^O((5), d) &= \frac{2(7d+12)}{d(d+1)(d+2)(d+4)(d+6)(d+8)(d-1)(d-2)(d-3)(d-4)}. \\
\text{Wg}^O((4, 1), d) &= \frac{-5d+4}{d(d+1)(d+2)(d+4)(d+8)(d-1)(d-2)(d-3)(d-4)}. \\
\text{Wg}^O((3, 2), d) &= \frac{-2(d^2 + 7d + 36)}{d(d+1)(d+2)(d+4)(d+6)(d+8)(d-1)(d-2)(d-3)(d-4)}. \\
\text{Wg}^O((3, 1^2), d) &= \frac{2(d^3 + 8d^2 + d - 36)}{d(d+1)(d+2)(d+4)(d+6)(d+8)(d-1)(d-2)(d-3)(d-4)}. \\
\text{Wg}^O((2^2, 1), d) &= \frac{d^2 + 3d + 4}{d(d+1)(d+2)(d+4)(d+8)(d-1)(d-2)(d-3)(d-4)}. \\
\text{Wg}^O((2, 1^3), d) &= \frac{-d^4 - 10d^3 - 7d^2 + 86d + 24}{d(d+1)(d+2)(d+4)(d+6)(d+8)(d-1)(d-2)(d-3)(d-4)}. \\
\text{Wg}^O((1^5), d) &= \frac{d^5 + 11d^4 + 5d^3 - 175d^2 - 122d + 408}{d(d+1)(d+2)(d+4)(d+6)(d+8)(d-1)(d-2)(d-3)(d-4)}.
\end{aligned}$$

$$\begin{aligned}
\text{Wg}^O((6), d) &= \frac{-2(21d^2 + 118d + 172)}{d(d+1)(d+2)(d+3)(d+4)(d+6)(d+8)(d+10)(d-1)(d-2)(d-3)(d-4)(d-5)}. \\
\text{Wg}^O((5, 1), d) &= \frac{2(7d^3 + 12d^2 - 35d - 10)}{d^2(d+1)(d+2)(d+3)(d+4)(d+6)(d+10)(d-1)(d-2)(d-3)(d-4)(d-5)}. \\
\text{Wg}^O((4, 2), d) &= \frac{5d^4 + 61d^3 + 406d^2 + 840d + 640}{d^2(d+1)(d+2)(d+3)(d+4)(d+6)(d+8)(d+10)(d-1)(d-2)(d-3)(d-4)(d-5)}. \\
\text{Wg}^O((4, 1^2), d) &= \frac{-5d^5 - 66d^4 - 131d^3 + 642d^2 + 1272d - 640}{d^2(d+1)(d+2)(d+3)(d+4)(d+6)(d+8)(d+10)(d-1)(d-2)(d-3)(d-4)(d-5)}. \\
\text{Wg}^O((3^2), d) &= \frac{4(d^4 + 13d^3 + 117d^2 + 300d - 240)}{d^2(d+1)(d+2)(d+3)(d+4)(d+6)(d+8)(d+10)(d-1)(d-2)(d-3)(d-4)(d-5)}. \\
\text{Wg}^O((3, 2, 1), d) &= \frac{2(-d^4 - 7d^3 - 24d^2 + 12d + 60)}{d^2(d+1)(d+2)(d+3)(d+4)(d+6)(d+10)(d-1)(d-2)(d-3)(d-4)(d-5)}. \\
\text{Wg}^O((3, 1^3), d) &= \frac{2(d^6 + 16d^5 + 49d^4 - 200d^3 - 810d^2 - 96d + 960)}{d^2(d+1)(d+2)(d+3)(d+4)(d+6)(d+8)(d+10)(d-1)(d-2)(d-3)(d-4)(d-5)}. \\
\text{Wg}^O((2^3), d) &= \frac{-(d^5 + 16d^4 + 101d^3 + 394d^2 + 2408d + 3840)}{d^2(d+1)(d+2)(d+3)(d+4)(d+6)(d+8)(d+10)(d-1)(d-2)(d-3)(d-4)(d-5)}. \\
\text{Wg}^O((2^2, 1^2), d) &= \frac{d^6 + 17d^5 + 77d^4 + 7d^3 - 446d^2 - 472d - 1280}{d^2(d+1)(d+2)(d+3)(d+4)(d+6)(d+8)(d+10)(d-1)(d-2)(d-3)(d-4)(d-5)}. \\
\text{Wg}^O((2, 1^4), d) &= \frac{-d^5 - 22d^4 - 154d^3 - 316d^2 + 339d + 1146}{d(d+1)(d+2)(d+3)(d+4)(d+6)(d+8)(d+10)(d-1)(d-2)(d-3)(d-5)}. \\
\text{Wg}^O((1^6), d) &= \frac{d^8 + 19d^7 + 68d^6 - 490d^5 - 2687d^4 + 1807d^3 + 17754d^2 + 6120d - 15360}{d^2(d+1)(d+2)(d+3)(d+4)(d+6)(d+8)(d+10)(d-1)(d-2)(d-3)(d-4)(d-5)}.
\end{aligned}$$

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BENOÎT COLLINS: DÉPARTEMENT DE MATHÉMATIQUE ET STATISTIQUE, UNIVERSITÉ D'OTTAWA, 585 KING EDWARD, OTTAWA, ON, K1N6N5 CANADA AND CNRS, INSTITUT CAMILLE JORDAN UNIVERSITÉ LYON 1, 43 BD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE, FRANCE
E-mail: bcollins@uottawa.ca

SHO MATSUMOTO: GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA, 464-8602, JAPAN. E-mail: sho-matsumoto@math.nagoya-u.ac.jp